Linear Homogeneous Differential Equation of Higher Order

[illegible] A diff. eqⁿ of the form $x^n \cdot \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots\dots + P_{n-1} x \frac{dy}{dx} + P_n y = X$, is known as linear homogeneous differential equation of order n where $P_1, P_2, \dots, P_n$ are constant and X is a function of x. ⟶ (1)

Ex. $x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + 5y = x^3$.

$x^2 \frac{d^2y}{dx^2} + y = \log x$.

NOTE

Since, eqⁿ (1) is linear homogeneous

Then putting $x = e^z$ i.e. $\log x = z$.

Now,

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{1}{x} \cdot \frac{dy}{dz}$$

$$= \frac{1}{x} \cdot Dy \text{, where } D = \frac{d}{dz}.$$

$$\therefore x \frac{dy}{dx} = Dy.$$

Again

$$\frac{d^2y}{dx^2} = \frac{1}{x} \cdot D^2y \cdot \frac{dz}{dx} - \frac{1}{x^2} \cdot Dy$$

$$= \frac{1}{x^2} D^2y - \frac{1}{x^2} Dy.$$

$$= \frac{1}{x^2} D(D-1)y.$$

$$\therefore x^2 \frac{d^2y}{dx^2} = D(D-1)y. \quad \text{etc}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x} \frac{dy}{dz}\right)$$

$$= \frac{1}{x} \cdot \frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{1}{x^2} Dy$$

$$= \frac{1}{x} \cdot \frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2} Dy$$

$$= \frac{1}{x^2} \cdot D \cdot Dy - \frac{1}{x^2} Dy$$

$$= \frac{1}{x^2}(D^2 - D)y$$

$$= \frac{1}{x^2} D(D-1)y$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y.$$

Then with this values the eqⁿ (1) reduces to linear diff. eqⁿ.

Solve: $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x$. $\rightarrow$ (i)

Eq^n (i) is 2nd order linear homogeneous.

$\therefore$ Putting $x = e^z$, the above eq^n reduces to

$$D(D-1)y - Dy + y = 2z, \quad D = \frac{d}{dz}$$

or, $(D^2 - 2D + 1)y = 0$. $\leftarrow$ (ii)

Eq^n (i) is 2nd order linear

Let, $y = e^{mz} (\neq 0)$ be a ~~trial~~ trivial sol^n of

$$(D^2 - 2D + 1)y = 0$$

$\therefore$ its A.E is, $(m^2 - 1)^2 = 0$.

or, $m = 1, 1$

$\therefore$ C.F $= (c_1 + c_2 z)e^z$

$= (c_1 + c_2 \log x)x$.

$$P.I = \frac{1}{D^2 - 2D + 1} \cdot 2z$$

$= (D^2 - 2D + 1)^{-1} \cdot 2z$

$= [1 - (D^2 - 2D) + (D^2 - 2D)^2 - \cdots] \cdot 2z$

$= 2z + 4$

$= 2(z + 2)$.

$= 2(\log x + 2)$

$\therefore$ general sol^n, $y = (c_1 + c_2 \log x)x + 2\log x + 4$.

(Ans)